LARGE N_C MEANS $N_C = 3$

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1 Introduction

It is well known[1] that the interacting part of the SU(2) linear σ model (LσM) Lagrangian density (after the spontaneous symmetry breaking shift) is:

$$\mathcal{L}^{int}_{L\sigma M} = g\overline{\psi}(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\psi \ + \ g'\sigma(\sigma^2+\vec{\pi}^2) - \ \lambda(\sigma^2+\vec{\pi}^2)^2/4, \tag{1}$$

where the couplings g, g', λ obey the chiral LσM relations

$$g = m_q/f_\pi, \qquad\qquad g' = m_\sigma^2/2f_\pi = \lambda f_\pi. \tag{2}$$

Here $f_\pi \approx 93$ MeV, and we take the fermion spinors as quark fields. In the chiral limit (CL) a once-subtracted dispersion relation combined with unitarity predicts[2]

$$1 \ - \ \frac{f_\pi^{CL}}{f_\pi} \ = \ \frac{m_\pi^2}{8\pi^2 f_\pi^2} \ \approx \ 0.03, \tag{3}$$

so the observed pion decay constant $f_\pi \approx 93$ MeV requires $f_\pi^{CL} \approx 90$ MeV. While the chiral relations (2) already hold at tree level[1], they also remain valid at one-loop level along with two additional LσM relations[3]:

$$m_\sigma = 2m_q, \qquad\qquad g = g_{\pi qq} = g_{\sigma qq} = 2\pi/\sqrt{N_c}. \tag{4}$$

The former relation in (4) is the famous NJL relation[4] and the latter equation in (4) for $N_c = 3$ gives $g = 2\pi/\sqrt{3} = 3.6276$, so the quark-level Goldberger-Treiman relation (GTR) in (2) predicts

$$m_q^{CL} \ = \ f_\pi^{CL} g \ \approx \ (90 \text{ MeV})2\pi/\sqrt{3} \ \approx \ 325 \text{ MeV}, \tag{5}$$

reasonably near the expected nonstrange constituent quark mass $m_q \approx M_N/3 \approx 313$ MeV. Also, the 3 constituent quarks in a nucleon (uud or ddu) then find the predicted πqq coupling $g \ = \ 2\pi/\sqrt{3} \ \approx \ 3.63$ to be nearby

$$g \ = \ g_{\pi NN}/3g_A \ \approx \ 3.47, \tag{6}$$

for the measured $g_{\pi NN} \approx 13.2$, $g_A \approx 1.267$ and near the GTR value $g = m_q/f_\pi^{CL} \approx 313$ MeV/ 90 MeV $\approx$ 3.48. Moreover, $g \ = \ 2\pi/\sqrt{3}$ in Eqs. (4), (5), (6) also follows from the Z = 0 compositeness condition[5,6].

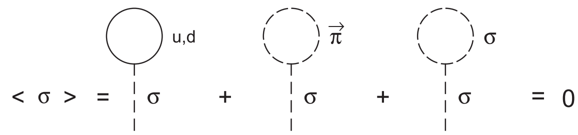

Figure 1. *Ben Lee null tadpole sum.*

2 Color Number $N_c = 3$ in the SU(2) LσM

It is well understood that the quark model must have an additional quantum number (color) in order to describe qqq decuplet as well as qqq octet baryons. The phenomenological determination of N_c is from the axial anomaly[7] (plus PCAC) or from the PVV LσM u,d quark loop[8]. Then the $\pi^0 \to 2\gamma$ decay rate of 7.7 eV implies that the amplitude magnitude satisfies

$$|M_{\pi^0\gamma\gamma}| \;=\; \frac{N_c}{3}\frac{\alpha}{\pi f_\pi} \;=\; (0.025 \pm 0.001)\ \text{GeV}^{-1}. \tag{7}$$

But $\alpha/\pi f_\pi \approx 0.0025$ GeV^{-1} for $f_\pi \approx 93$ MeV, so that Eq. (7) suggests $N_c/3 = 1$ or $N_c = 3$.

The SU(2) LσM theoretical value of N_c stems from the Ben Lee null tadpole sum[9] (due to u,d quark loop, $\vec{\pi}$ loop and σ loop tadpole graphs linked to the scalar σ meson as shown in Fig. 1):

$$<\sigma> = \; -8N_c m_q g \int \frac{d^4p}{p^2 - m_q^2} \;+\; 3g' \int \frac{d^4p}{p^2 - m_q^2} \;=\; 0. \tag{8}$$

Here we have dropped the small pion loop (due to the Nambu-Goldstone theorem), used $8 = 4 \times 2$ with 4 from the fermion trace, 2 from u,d flavours, and invoked the combinatoric factor of 3 for the $\sigma - \sigma - \sigma$ coupling. The false vacuum[1] has $<\sigma>= f_\pi \neq 0$, $<\pi>=0$, whereas after the shift the true vacuum $<\sigma>=<\pi>=0$ is characterized by Fig. 1 and by Eq. (8). We solve Ben Lee's equation (8) above by using the LσM chiral relations $g = m_q/f_\pi$ and $g' = m_\sigma^2/2f_\pi$ from Eq. (2) together with simple dimensional analysis requiring quadratic divergence mass scales

$$\int \frac{d^4p}{p^2 - m_q^2} \sim m_q^2, \qquad \int \frac{d^4p}{p^2 - m_\sigma^2} \sim m_\sigma^2. \tag{9}$$

Thus Eq. (8) above implies[3]

$$N_c(2m_q)^4 \;=\; 3m_\sigma^4, \tag{10}$$

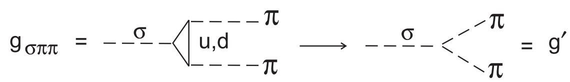

Figure 2. *LσM cubic meson log-divergent coupling* $g_{\sigma\pi\pi} \to g'$.

so the NJL value $m_\sigma = 2m_q$ requires $N_c = 3$ in Eq. (10). The SU(2) LσM also requires the loop value $m_\sigma = 2m_q$, which we derive in the next section, and again $N_c = 3$ follows from the LσM null tadpole condition Eq. (8) leading to Eq. (10).

3 Log-Divergent Gap Equation (LDGE) in the LσM

Defining the pion decay constant via $< 0\, |A^i_\mu|\pi^j >= i f_\pi q_\mu \delta^{ij}$, the u,d quark loops combined with the GTR $f_\pi g = m_q$ leads to the LDGE[10] in the CL :

$$-i4N_c g^2 \int \frac{\bar{d}^4 p}{(p^2 - m_q^2)^2} \;=\; 1. \tag{11}$$

It is a gap equation because the "mass gap" m_q cancels out to form (11). This LDGE Eq. (11) also follows from the NJL analysis. Morever, the u,d quark loops for the pion form factor (using $g\gamma_5$ pseudoscalar scaling) in turn requires $F_\pi(q^2 = 0) = 1$ since then PVV quark loops predict the pion form factor[11]

$$F_\pi(q^2) \;=\; -i4N_c g^2 \int_0^1 dx \int \frac{\bar{d}^4 p}{[p^2 - m_q^2 + x(1-x)q^2]^2} \quad , \tag{12}$$

for $\bar{d}^4 p = d^4 p/(2\pi)^4$. Clearly Eq. (12) reduces to the LDGE Eq. (11) when $F_\pi(q^2 = 0) = 1$.

Next, the 3-point function $g_{\sigma\pi\pi}$ quark triangle in Fig. 2 reduces in the CL to[3]

$$g_{\sigma\pi\pi} \;=\; -8ig^3 N_c m_q \int \frac{\bar{d}^4 p}{[p^2 - m_q^2]^2} \;=\; 2g m_q \quad , \tag{13}$$

by virtue of the LDGE (11) and then the GTR, $m_q = f_\pi g$ "shrinks" (13) to

$$g_{\sigma\pi\pi} \;=\; 2m_q g \;=\; (2m_q)^2/2f_\pi^{CL} \;=\; m_\sigma^2/2f_\pi \;=\; g', \tag{14}$$

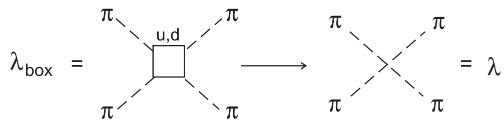

Figure 3. *LσM quartic meson log-divergent coupling $\lambda_{box} \to \lambda$.*

the tree-level LσM value in Eq. (2). This is valid *only if* $m_\sigma = 2m_q$ for the loop-level LσM [3].

Also, the 4-point function $g_{\pi\pi\pi\pi}$ quark box in Fig. 3 shrinks in the CL to[3]

$$\lambda_{box} \; = \; -8iN_c g^4 \int \frac{\bar{d}^4 p}{[p^2 - m_q^2]^2} \; = \; 2g^2 \; = \; g'/f_\pi \; = \; \lambda \; , \qquad (15)$$

the LσM contact term in Eq. (2), but again we must invoke the LDGE (11) and $m_\sigma = 2m_q$. In effect, the quark loops of Figs. 1,2,3 dynamically reproduce the entire SU(2) LσM Lagrangian with $N_c = 3$ and $g = 2\pi/\sqrt{3}$, $g' = 2g\, m_q$, and $\lambda = 2g^2 = 8\pi^2/3$. Note that the large N_c limit for a $U_L(1) \times U_R(1)$ model predicts[12] $m_\sigma \to 2m_q$ but $\lambda \to g^2$ (not $2g^2$) as $N_c \to \infty$.

4 Chiral phase transition

An independent test of the above SU(2) LσM stems from the chiral phase transition approach, where the thermalized fermion (quark) propagator is replaced at finite temperature by[13]

$$\frac{i(\gamma \cdot p + m)}{p^2 - m^2} \; \to \; \frac{i(\gamma \cdot p + m)}{p^2 - m^2} \; - \; \frac{2\pi\delta(p^2 - m^2)(\gamma \cdot p + m)}{e^{|P_0/T|} + 1} \quad . \qquad (16)$$

This approach leads to the chiral restoration (melting) temperature T_c, which corresponds to a second-order phase transition. Over the past 15 years, many authors have predicted for two quark flavours that this order parameter is[14]

$$T_c \; = \; 2f_\pi^{CL} \; = \; 2 \times 90 \text{ MeV} \; = \; 180 \text{ MeV} \quad , \qquad (17)$$

where the 3% reduction of f_π in Eq. (3) above has been used. Only recently, F. Karsch[15] has "measured" T_c via the computer lattice to find for $N_f = 2$:

$$T_c \; = \; 173 \; \pm \; 8 \text{ MeV} \quad , \qquad (18)$$

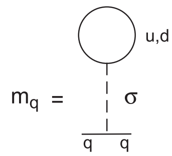

Figure 4. *LσM quark tadpole representation of quark mass m_q.*

so the various melting studies theoretically predicting the nearby Eq. (17) should be reviewed in detail.

4.1 Melting the SU(2) Quark Mass

As T increases to the chiral "melting" temperature T_c, the quark mass melts, and the σ tadpole graph of Fig. 4 via Eq. (16) gives[16]

$$0 \leftarrow m_q(T_c) \;=\; m_q \;+\; \frac{8N_c g^2 m_q}{-m_\sigma^2}\left(\frac{T_c^2}{2\pi^2}\right) J_+ \quad , \tag{19}$$

where we use $N_c = 3$ and the integral $J_+ = \int_0^\infty x dx [e^x + 1]^{-1} = \pi^2/12$. Then Eq. (19) reduces to

$$T_c^2 \;=\; m_\sigma^2 / g^2 \quad . \tag{20}$$

Note that the NJL-LσM relation $m_\sigma \;=\; 2m_q$ combined with the GTR $g = m_q/f_\pi^{CL}$ means that the square root of (20) is $T_c = 2f_\pi^{CL}$, Eq. (17) above.

4.2 Melting the Scalar σ Mass in the LσM

Assuming the Lagrangian density quartic coupling is proportional to $\mathcal{L} \sim -(\lambda/4)\sigma^4$, but with $J_+ \to J_- = \int_0^\infty x dx [e^x - 1]^{-1} = \pi^2/6$, Refs. [14] then find

$$T_c^2 \;=\; 2m_\sigma^2/\lambda \quad , \tag{21}$$

in an SU(2) LσM context. Then, independent of the value of T_c or m_σ, Eqs. (20), (21) together require $\lambda = 2g^2$, as already found in Eq. (15) in LσM loop order.

4.3 Melting the Quark Condensate in QCD

Noting that thermalizing the quark condensate as $\delta < \overline{q}q > = < \overline{q}q > \delta\overline{q} + < \overline{q}q > \delta q$, Ref. [17] finds in QCD

$$T_c^2 \ = \ 3m_q^2/\pi^2 \quad . \tag{22}$$

Expressing the square root of Eq. (22) together with Eq. (20) and again using the NJL-LσM relation $m_\sigma = 2m_q$ leads to the on-shell meson-quark coupling (with $N_c = 3$)

$$g \ = \ 2\pi/\sqrt{3}, \tag{23}$$

as already predicted in Sec. 1 for the LσM.

In QCD language, $\alpha_s(1 \text{ GeV}) \approx 0.5$[18] grows at $m_\sigma \sim 600$ MeV to $\alpha_s(m_\sigma)|_{\text{QCD}} \approx \pi/4$, the infrared limit[19]. Then the C_{2F} factor increases α_s to $\alpha^{\text{eff}}_{\text{QCD}}(m_\sigma) = (4/3)\pi/4 = \pi/3$, i.e., to quark condensation with $\alpha^{\text{eff}}_s \approx 1$. In L$\sigma$M language Eq. (23) corresponds to $g^2/4\pi = \pi/3$, the above infrared value of QCD[20]. Note that the latter has already been derived in Eq. (4) via the loop-order LσM [3] or via the Z = 0 compositeness condition[5].

5 Invoking the Dim. Reg. Lemma for Loop-Order LσM

Lastly, we can self-consistently nonperturbatively derive the NJL-LσM relation $m_\sigma = 2m_q$. The constituent quark mass σ tadpole graph of Fig. 4 generates the quadratically divergent mass gap equation (QDGE)

$$m_q \ = \ -\frac{8iN_c g^2}{m_\sigma^2}\int \frac{\bar{d}^4p \ m_q}{p^2 \ - \ m_q^2} \quad . \tag{24}$$

Such a quadratic divergence also appears in the dimensional regularization (dim. reg.) lemma (DRL)

$$\int \bar{d}^4p\left[\frac{m_q^2}{(p^2-m_q^2)^2} - \frac{1}{p^2-m_q^2}\right] = \lim_{l\to 2}\frac{im_q^{2l-2}}{(4\pi)^2}[\Gamma(2-l)+\Gamma(1-l)] = -\frac{im_q^2}{(4\pi)^2}, \tag{25}$$

due to the gamma function identity[3] $\Gamma(2-l)+\Gamma(1-l) \to -1$ as $2l \to 4$. Combining the DRL Eq. (25) with the QDGE Eq. (24) and the LDGE Eq. (11) and dividing out the nonzero quark mass (gap), one obtains[3]

$$1 \ = \ \frac{2m_q^2}{m_\sigma^2}\left[1 \ + \ \frac{g^2N_c}{4\pi^2}\right] \quad . \tag{26}$$

Further, solving Eq. (26) as $1 = 1/2 + 1/2$ (due to the pion and σ bubbles giving $m_\pi^2 = 0,\ m_\sigma^2 = N_c g^2 m^2/\pi^2$), one finds[3] the NJL-LσM and Z = 0 loop-order solutions of Eqs. (4). The above dim. reg. formalism is

a) compatible with the LDGE "nonperturbative shrinkage" of the quark triangle and box LσM Eqs. (14) and (15) for any value of N_c, giving again $m_\sigma = 2m_q,\ g = 2\pi/\sqrt{N_c}$,

b) compatible with the chiral phase transition approach of Sec. 4, but only when $N_c = 3$,

c) compatible with analytic, zeta function, and Pauli-Villars regularization schemes when the massless quadratic loop analogue of Eqs. (9) holds[21]:

$$\int d^4p/p^2 \ = \ 0 \ , \tag{27}$$

again due to simple dimensional analysis.

Recall that the massive SU(2) loop versions Eqs. (9) allowed us to theoretically deduce that $N_c = 3$ due to the B.W. Lee null tadpole sum Eq. (8), leading to Eq. (10) because we already know $m_\sigma = 2m_q$ in the NJL or LσM theories at loop order.

6 Conclusion

Prof. 't Hooft began his film lectures by first writing the LσM Lagrangian, but instead let $N_c \to \infty$ in order to perturbatively solve the LσM and also QCD[22]. While we have no objection to such an approach, we suggest it in fact is possible to start with the usual value $N_c = 3$ and instead nonperturbatively solve in loop order the SU(2) LσM with the NJL–Ben Lee null tadpole solution $m_\sigma = 2m_q$. This also leads to LσM nonperturbative shrinkage in Sec. 3 via the LDGE, to the chiral phase transition temperature $T_c = 2f_\pi^{CL} \approx 180$ MeV for $N_f = 2$, and to the formal LDGE-DRL-QDGE solution $m_\sigma = 2m_q,\ g = 2\pi/\sqrt{N_c}$ in Sec. 4. Of course the LσM requires $g^2 N_c =$ constant $(4\pi^2)$ with amplitudes scaling like $1/N_c^2$ (1/9), so 't Hooft's large N_c result that $1/N_c^2$ terms vanish as $N_c \to \infty$ is 90% true anyway when $N_c = 3$.

Acknowledgements

The author thanks R. Delbourgo, J. Cleymans, V. Elias and P. Zenczykowski for prior discussions.

References

1. M. Gell Mann and M. Levy, Nuovo Cimento **16** (1960) 705; V. de Alfaro, S. Fubini, G. Furlan and C. Rossetti, *Currents in Hadron Physics* (North Holland, 1973), Chap. 5.
2. M.D. Scadron, Repts. on Prog. in Phys. **44** (1981) 213; S.A. Coon and M.D. Scadron, Phys. Rev. C **23** (1981) 1150.
3. R. Delbourgo and M.D. Scadron, Mod. Phys. Lett. A **10** (1995) 251; M.D. Scadron, *Sigma Meson* Workshop, Kyoto June 2000 [*hep-ph*/0007184].
4. Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122** (1961) 345.
5. A. Salam, Nuovo Cimento **25** (1962) 224; S. Weinberg, Phys. Rev. **130** (1963) 776; M.D. Scadron, *ibid.*, D **57** (1998) 5307.
6. V. Elias and M.D. Scadron, Phys. Rev. Lett. **53** (1984) 1129; also see M.D. Scadron, Ann. Phys. **148** (1983) 257.
7. S.L. Adler, Phys. Rev. **177** (1969) 2426; J. Bell and R. Jackiw, Nuovo Cimento **60** (1969) 47.
8. R. Delbourgo, D. Lui and M.D. Scadron, Int. J. Mod. Phys. A**14** (1999) 4331.
9. B.W. Lee, *Chiral Dynamics* (Gordon and Breach, 1972), p. 12.
10. T. Hakioglu and M.D. Scadron, Phys. Rev. D**42** (1990) 941; D**43** (1991) 2439.
11. N. Paver and M.D. Scadron, Nuovo Cimento A**78** (1983) 159; A. Bramon, Riazuddin and M.D. Scadron, J. Phys. G **24** (1998) 1.
12. J. Zinn-Justin, Nucl. Phys. B**367** (1991) 105; K. Akama, Phys. Rev. Lett. **76** (1996) 184.
13. L. Dolan and R. Jackiw, Phys. Rev. D**9** (1974) 3320; J. Kapusta, *Finite Temperature Field Theory* (Cambridge Univ. Press, U.K. 1989).
14. D. Bailin, J. Cleymans and M.D. Scadron, Phys. Rev. D**31** (1986) 164; J. Cleymans, A. Kocic amd M.D. Scadron, *ibid.*, D**39** (1989) 323. Also see review by T. Hatsuda, Nucl. Phys. A**544** (1992) 27.
15. F. Karsch, *Quark Matter Conf. 2001*, Stony Brook, Jan. 2001 [*hep-ph*/0103314]; F. Karsch et al., Nucl. Phys. B**605** (2001) 579.
16. N. Bilic, J. Cleymans and M.D. Scadron, Int. J. Mod. Phys. A**10** (1995) 1169.
17. M.D. Scadron and P. Zenczykowski, Invited talk at *Chiral Fluctuations in Hadronic Matter* Workshop [*hep-th*/0106154].
18. See, e.g., A. de Rujula, H. Georgi and S. Glashow, Phys. Rev. D**12** (1975) 147.
19. A. Mattingly and P. Stevenson, Phys. Rev. Lett. **69** (1992) 1320.
20. V. Elias and M.D. Scadron, Ref. [6] and Phys. Rev. D**30** (1984) 647;

L. Baboukhadia, V. Elias and M.D. Scadron, J. Phys. G**23** (1997) 1065.
21. R. Delbourgo, A. Rawlinson and M.D. Scadron, Mod. Phys. Lett. A**13** (1998) 1893.
22. G. 't Hooft, Nucl. Phys. B**72** (1974) 461.